

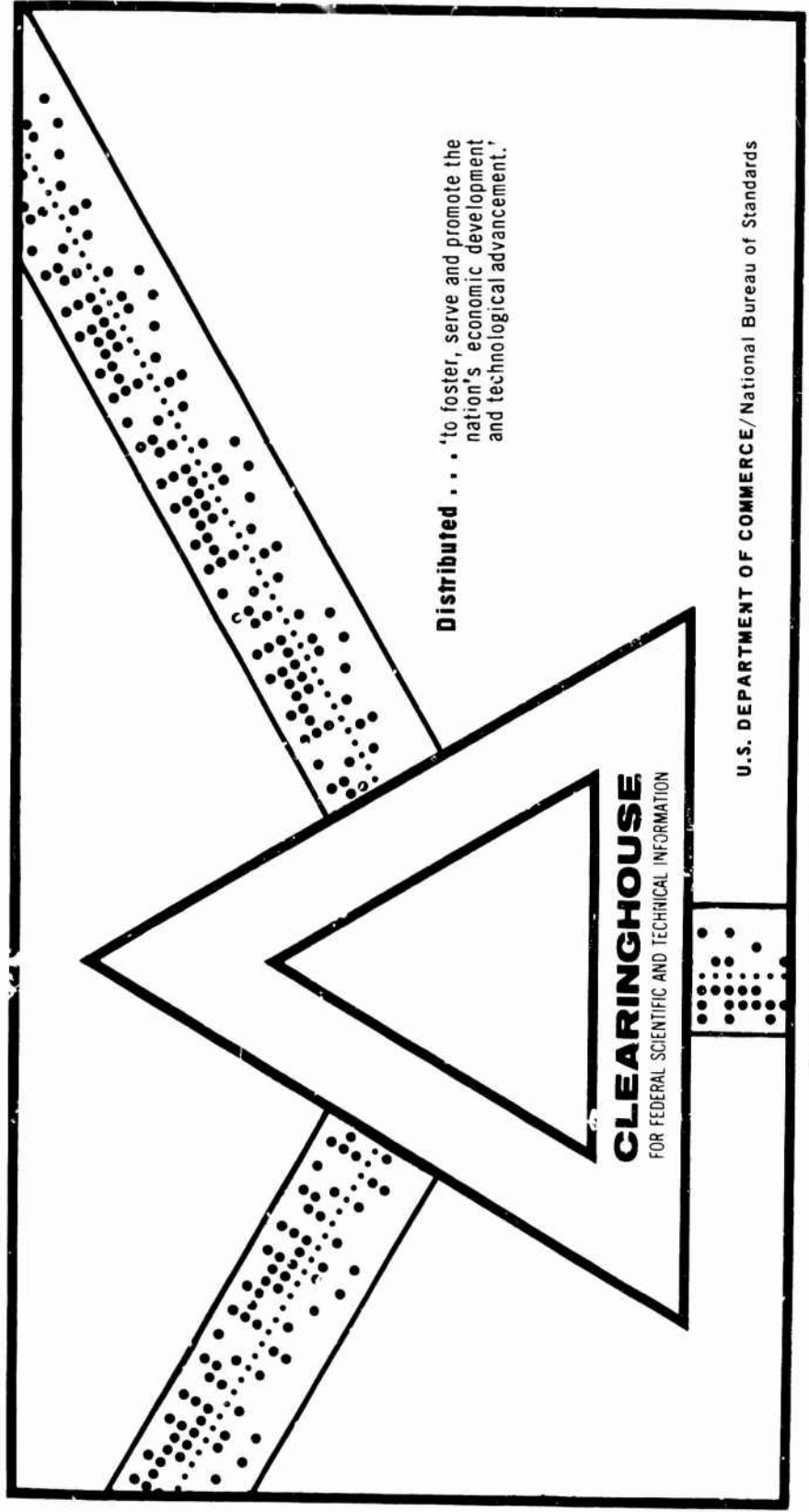
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PROBABILITY OF DEFECTIVE ASSEMBLIES WHEN COMPONENT TOLERANCES  
ARE INCORRECT

Lance W. Jayne, et al

Army Natick Laboratories  
Natick, Massachusetts

December 1969



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TECHNICAL REPORT  
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WHEN COMPONENT TOLERANCES ARE INCORRECT

by

Lance Y. Janna

and

Edward A. Post, Jr.

D D C  
RESEARCH  
REPORT

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PROBABILITY OF DEFECTIVE ASSEMBLIES  
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## FOREWORD

A major problem in all engineering design is that of reliability and quality control. Standard procedures exist for assuring the desired reliability under ordinary circumstances. This report describes an unusual situation, in which the tolerances on the components of an assembly are not small enough to ensure that the assembly will work properly, and provides estimates of the probability of a defective assembly in this case.

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## ABSTRACT

This report describes an investigation of how errors in components of an assembly can affect its performance. In particular the report deals with the situation, uncommon in engineering practice, where the output tolerance of the assembly may be violated even though the tolerances on the components are all met. This situation is analyzed to estimate the probability that the output tolerance will be satisfied given that the component tolerances are met. Three methods are described for estimating this probability, their results are compared in a number of cases, and a best method is chosen. Several simple rules, suitable for preliminary estimates, are also given. An example is worked out showing a simple application of the method.

## 1. INTRODUCTION

This report deals with certain aspects of the general problem of errors and tolerances in the design and testing of equipment. It is presumed that the piece of equipment is required to operate at a certain level of output. Ordinarily the designer assigns a certain error-tolerance to this output, chosen so that the equipment will function properly if the output error satisfies its tolerance. The output error usually arises from errors in the various components that have been assembled to make the piece of equipment. The designer will customarily know the relation between the output error and the component errors. Common practice (see Bowker and Lieberman<sup>(1)</sup>) is that the designer will combine this relation with the output tolerance to find tolerances on each component such that satisfaction of these component tolerances will ensure that the output tolerance is met.

We are concerned here with the uncommon situation where satisfaction of the component tolerances does not ensure satisfaction of the output tolerance. This state of affairs can arise when an error has been made in choosing the component tolerances, or when it is impractical (or too expensive) to make the component tolerances small enough. In either case we must face the possibility that all the components will meet their tolerances but some of the assembled pieces of equipment will not work properly. The practical information that we want is the probability that the output tolerance will



be satisfied. With this information we can estimate how many extra pieces of equipment must be manufactured on the average in order to obtain a given number of workable assemblies.

In the following section we shall describe the general procedure for estimating the probability that the output satisfies its tolerance, supposing that each component error is normally distributed with zero mean, known variance and known tolerance. Three mathematical methods are given for carrying out the calculations. One is of Monte-Carlo type and is described in Section 3. The other two methods use the Characteristic Function in different ways. Section 4 gives formulas for the Characteristic Functions of the various distributions, and Sections 5 and 6 use these formulas in estimating the desired probability. Various simple approximations and limiting cases are examined in Section 7. Section 8 describes the results, which are then discussed in Section 9. The report closes with a simple example of how these estimates might be used in practice, Section 10.

We let  $Y_j$  be the error in the  $j$ -th component,  $j = 1, 2, \dots, N$ , and  $X_o$  is the error in the output. The relation between the output error and the component errors is taken as linear,

$$X_o = \sum_{j=1}^N C_j Y_j \quad (1)$$

where the  $C_j$  are assumed to be known constants.

It is assumed initially that  $Y_j$  is normally distributed with zero mean and variance  $\sigma_j^2$ . Then we may define

$$\left. \begin{aligned} X_j &= C_j Y_j \\ S_j &= |C_j| \sigma_j \end{aligned} \right\} \quad j = 1, 2, \dots, N \quad (2)$$

and the relation (1) can be written

$$X_o = \sum_{j=1}^N X_j \quad (3)$$

where  $X_j$  is normally distributed with zero mean and variance  $S_j^2$ .

We let  $D_j > 0$  be the tolerance on the error,  $Y_j$ , in the  $j$ -th component, and  $B_o > 0$  be the tolerance on the output error,  $X_o$ . Thus, when  $Y_j$  satisfies its tolerance, we have

$$|Y_j| \leq D_j$$

and, if  $X_0$  satisfied its tolerance, then

$$|X_0| \leq B_0$$

We define also

$$B_j = |C_j| D_j \tag{4}$$

as the tolerance on  $X_j$ , so that, if  $X_j$  satisfied its tolerance, then

$$|X_j| \leq B_j$$

We notice also that

$$B_j / S_j = D_j / \sigma_j \tag{5}$$

We now define certain probabilities.  $P_j$  is the probability that the  $j$ -th error satisfies its tolerance, i.e.,

$$P_j = \text{prob}[|Y_j| \leq D_j] = \text{prob}[|X_j| \leq B_j] \tag{6}$$

$P_c$  is the probability that all component errors satisfy their tolerances. We assume that the component errors are independent of each other, and therefore

$$\begin{aligned} P_c &= \text{prob}[|X_j| \leq B_j \text{ for all } j] \\ &= \prod_{j=1}^N P_j = P_1 P_2 \dots P_N \end{aligned} \tag{7}$$

Further, we define

$P_0 = \text{prob } \left[ \begin{array}{l} \text{the output error satisfies its tolerance and all component} \\ \text{errors satisfy their tolerances} \end{array} \right]$

$$P_0 = \text{prob } \left[ |X_0| \leq B_0 \text{ and } |X_j| \leq B_j \text{ for all } j \right] \quad (8)$$

The theorem on compound probability asserts that

$$P_0 = \text{prob } \left[ |X_0| \leq B_0, \text{ given that } |X_j| \leq B_j \text{ for all } j \right] \\ \times \text{prob } \left[ |X_j| \leq B_j \text{ for all } j \right] \quad (9)$$

The probability that is of greatest practical interest is

$$P^* = \text{prob } \left[ |X_0| \leq B_0, \text{ given that } |X_j| \leq B_j \text{ for all } j \right] \quad (10)$$

Then we can write (9) with the aid of (7) and (10) as

$$P^* = P_0 / P_c = P_0 / \prod_{j=1}^N P_j \quad (11)$$

Finally, it is useful to define

$$\Delta = \sum_{j=1}^N B_j = \sum_{j=1}^N |C_j| D_j \quad (12)$$

If all the components satisfy their tolerances, then, using a familiar property of inequalities, we find

$$|X_0| = \left| \sum_{j=1}^N X_j \right| \leq \sum_{j=1}^N |X_j| \leq \sum_{j=1}^N B_j$$

and so, because of (12),  $X_0$  must satisfy the inequality

$$|X_0| \leq \Delta \quad (13)$$

If  $\Delta \leq B_0$  then (13) implies

$$|X_0| \leq \Delta \leq B_0$$

In this case we see that, if each component satisfies its tolerance, the output error  $X_0$  must always satisfy its tolerance, and from (10) and (11) we conclude that

$$P^* = 1, \quad P_0 = P_c$$

This case is the common one in design practice, i.e., the tolerances are set so that, if each component meets its tolerance, the output will necessarily satisfy its tolerance. However, in this paper we are interested in the opposite case, where

$$\Delta > B_0$$

and

$$0 \leq P^* \leq 1$$

Our main objective is to estimate  $P^*$ . We define  $F^*(X_0)$  as the density function of the output when the separate component errors all satisfy their tolerances. Since the component errors are normally distributed, their density functions, when they satisfy their tolerances, are symmetrically-truncated normal distributions, and  $F^*(X_0)$  is the density function of a finite sum of such distributions.  $P^*$  is the integral between  $-B_0$  and  $B_0$  of  $F^*(X_0)$ . Unfortunately  $F^*(X_0)$  is not easily expressible in terms of the parameters

of the component density functions. However, we can make a number of simple comments about the behavior of  $F^*(X_0)$ .

(i) If the component error tolerances are all very large, i.e.,  $B_j \gg S_j$ , each component error is approximately normally distributed, hence  $F^*(X_0)$  is approximately a normal function.

(ii) If  $N$ , the number of error components, is large, the Central Limit Theorem leads us to expect that  $F^*$  will be approximately a normal function.

(iii) Contrariwise,  $F^*$  will depart furthest from normality when some  $B_j/S_j$  are small and when  $N$  is small. This will be particularly so when one component dominates all the rest, so that effectively  $N = 1$ .



### 3. MONTE-CARLO METHOD

This method is based on counting the numbers of successes in sampling from distributions that are random, independent and normally distributed with zero mean. A computer program was written to carry out this procedure.

The program has as input the quantities  $C_j$ ,  $D_j$  ( $j = 1, 2, \dots, N$ ) and  $B_0$ , together with the list of random numbers.  $L$  sets of  $N$  random numbers are read in successively. For each set, the  $N$  random numbers are taken as the values of  $Y_j$  ( $j = 1, 2, \dots, N$ ), and the value of  $X_0$  is calculated from (1). Counts are made of the following quantities:

$n_j$  ( $j = 1, \dots, N$ ) is the number of cases for which  $|Y_j| \leq D_j$

$n_c$  is the number of cases for which  $|Y_j| \leq D_j$  for all  $j$ .

$n_0$  is the number of cases for which  $|Y_j| \leq D_j$  for all  $j$   
and  $|X_0| \leq B_0$

Then we obtain the estimates

$$P_j = n_j/L \quad j = 1, 2, \dots, N$$

$$P_c = n_c/L$$

$$P_0 = n_0/L$$

$$P^* = P_0/P_c = n_0/n_c.$$

In using this method we had to choose  $L$  large enough so that reasonably stable estimates of  $P^*$  were obtained. The choice  $L = 200$  was used, and the entire procedure repeated four times with different sets of random numbers. The final estimates of the probabilities are given as the means of the results for the four repetitions.

#### 4. CHARACTERISTIC FUNCTIONS

The remaining two methods of estimating  $P^*$  employ the Characteristic Function (or Fourier Transform) as the main tool in the analysis. In this section we present the general formulas that form the basis of these methods.

If  $F(X)$  is a density function, then

$$\phi(t) = \int_{X=-\infty}^{X=+\infty} e^{-itX} F(X) dX \quad (14)$$

is its Characteristic Function or Fourier Transform. It is unnecessary to dwell on the properties of the Characteristic Function which are well-known. We record only one formula, which is easily derived from the Complex Inversion Relation,

$$\int_{\alpha=-X}^X F(\alpha) d\alpha = \pi^{-1} \int_{t=-\infty}^{\infty} \phi(t) t^{-1} \sin(tX) dt \quad (15)$$

This formula expresses the area under the density curve between  $-X$  and  $X$  (or the probability that the variable lies between  $-X$  and  $X$ ) directly as an integral of  $\phi(t)$ .

We define  $F_j(X_j)$  as the density function for  $X_j$  and  $\phi_j(t)$  as the corresponding Characteristic Function. Similarly  $\phi_0(t)$  is the Characteristic Function of the output distribution,  $F^*(X_0)$ . Since the  $X_j$  are assumed to be independent, it is well-known that

$$\phi_0(t) = \prod_{j=1}^N \phi_j(t) \quad (16)$$

Both methods of estimating  $P^*$  are based on finding  $\phi_0(t)$  from the  $\phi_j(t)$  by means of (16).

To find  $\phi_j(t)$  we first write down the density function of  $X_j$ , which is that for a normal distribution with variance  $S_j^2$ , truncated at  $\pm B_j$ ,

$$F_j(X_j) = \begin{cases} \int_{u=-B_j}^{B_j} e^{-u^2/(2S_j^2)} du \end{cases}^{-1} e^{-X_j^2/(2S_j^2)}, & |X_j| \leq B_j \\ = 0 & |X_j| > B_j$$

This is then substituted into (14), and we find, after some manipulation,\*

$$\phi_j(t) = \frac{e^{-\rho_j^2}}{\operatorname{erf} \gamma_j} \operatorname{Re} \{ \operatorname{erf}(\gamma_j + i\rho_j) \} \quad (17)$$

where

$$\gamma_j = B_j / (S_j \sqrt{2}) > 0 \quad (18)$$

$$\rho_j = t S_j / \sqrt{2} \quad (19)$$

A series representation of  $\phi_j(t)$  in real terms may be derived by expanding the Error Function in a Taylor Series about  $\gamma_j$ . By means of (A.5) we find

$$\phi_j(t) = e^{-\rho_j^2} \left\{ 1 - \frac{2\pi^{-1/2} e^{-\gamma_j^2}}{\operatorname{erf} \gamma_j} \sum_{n=1}^{\infty} (-\rho_j^2)^n \frac{H_{2n-1}(\gamma_j)}{(2n)!} \right\} \quad (20)$$

---

\*For completeness a list of the basic formulas relating to the Error Function is given in Appendix A.

where  $H_k$  is the Hermite Polynomial of degree  $k$ . This series converges absolutely for any finite values of  $\rho_j$  and  $\gamma_j$ . Further, we may expand  $e^{-\rho_j^2}$  about  $\rho_j = 0$  and obtain explicitly the leading terms (up to  $t^4$ ) in the expansion of  $\phi_j(t)$  about  $t = 0$ ,

$$\begin{aligned} \phi_j(t) = & 1 - S_j^2 \left( 1 - \frac{2\pi^{-1/2} \gamma_j e^{-\gamma_j^2}}{\operatorname{erf} \gamma_j} \right) (t^2/2!) \\ & + S_j^4 \left\{ 3 - 2\pi^{-1/2} \frac{\gamma_j (3 + 2\gamma_j^2) e^{-\gamma_j^2}}{\operatorname{erf} \gamma_j} \right\} (t^4/4!) \\ & - \dots \end{aligned} \quad (21)$$

Although the series in (20) converges for any finite values of  $\rho_j$  and  $\gamma_j$ , the convergence is slow when  $\rho_j$  is large, and an alternative method of computation is needed. For this purpose it is convenient to use the real and imaginary parts,  $W_r$  and  $W_i$ , of the complex function,  $W$ , defined in (A.9). From (17), (A.10) and (A.11) we obtain the exact formula

$$\phi_j(t) = \frac{1}{\operatorname{erf} \gamma_j} \left\{ e^{-\rho_j^2} - e^{-\gamma_j^2} \left[ W_r(\rho_j + i\gamma_j) \cos(B_j t) - W_i(\rho_j + i\gamma_j) \sin(B_j t) \right] \right\} \quad (22)$$

A rational approximation for  $W$  is given in (A.12), and from it we may derive the following approximations for  $W_r$  and  $W_i$ :

$$W_r(\rho_j + i\gamma_j) = \sum_{k=1}^3 r_k \{ (-\gamma_j \alpha_{kj} + \rho_j B_j t) / (\alpha_{kj}^2 + B_j^2 t^2) \} \quad (23)$$

$$W_i(\rho_j + i\gamma_j) = \sum_{k=1}^3 r_k \{ (\rho_j \alpha_{kj} + \gamma_j B_j t) / (\alpha_{kj}^2 + B_j^2 t^2) \} \quad (24)$$

Here

$$\alpha_{kj} = \rho_j^2 - \gamma_j^2 - \eta_k, \quad (25)$$

and  $\rho_k$  and  $\eta_k$  are constants of the approximation, listed in Appendix A.



## 5. APPROXIMATION USING MOMENTS

This method consists of assuming that the output density function is approximately of the form

$$g(X_o) = S^{-1}(2\pi)^{-1/2} [G_0 + G_2(X_o/S)^2 + G_4(X_o/S)^4] e^{-X_o^2/(2S^2)} \quad (26)$$

where  $G_0$ ,  $G_2$  and  $G_4$  are constants to be determined, and  $S^2$  is the exact second moment of the output. The constants  $G_0$ ,  $G_2$  and  $G_4$  are chosen by matching moments, i.e., by using

$$\int_{-\infty}^{\infty} X_o^{2K} g(X_o) dX_o = M_{2K}, \quad K = 0, 1, 2 \quad (27)$$

where  $M_{2K}$  are the exact, even-ordered moments of the output distribution.

$M_0$ ,  $M_2$  and  $M_4$  are determined by using the well-known relation

$$M_{2K} = (-j)^{2K} \frac{d^{2K} \phi_o}{dt^{2K}}(0) \quad (28)$$

Because the  $\phi_j(t)$  are characteristic functions and are all even in  $t$ , we have

$$\phi_j(0) = 1, \quad \phi_j'(0) = \phi_j'''(0) = 0 \quad (29)$$

Differentiating (16) and using (29) we find

$$M_0 = \phi_0(0) = 1 \quad (30)$$

$$M_1 = -i \phi_0'(0) = 0 \quad (31)$$

$$M_2 = -\phi_0''(0) = \sum_{j=1}^N [-\phi_j''(0)] \quad (32)$$

$$M_3 = i \phi_0'''(0) = 0 \quad (33)$$

$$M_4 = \phi_0^{IV}(0) = 3[\phi_0''(0)]^2 + \sum_{j=1}^N [\phi_j^{IV}(0) - 3\{\phi_j''(0)\}^2] \quad (34)$$

From (21) we obtain

$$-\phi_j''(0) = S_j^2 \{1 - [z_j A'(z_j)/A(z_j)]\} \quad (35)$$

$$\phi_j^{IV}(0) = S_j^4 \{1 - [z_j A'(z_j)/A(z_j)](3 + z_j^2)\} \quad (36)$$

where

$$z_j = B_j/S_j = 2^{1/2} x_j \quad (37)$$

and the functions  $A$  and  $A'$  are defined in (A.2) and (A.6). We calculate  $M_2$  and  $M_4$  by inserting these values in (32) and (34).

If we combine (26) with (27) and use the general formula

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} v^{2n} e^{-v^2/(2S^2)} dv = S^{2n+1} \frac{n}{\pi} (2n-1), \quad n \geq 1 \quad (38)$$

we obtain the following linear, algebraic equations for

$G_0, G_2$  and  $G_4$ :

$$G_0 + G_2 + G_4 = 1$$

$$G_0 + 3G_2 + 15G_4 = M_2/S^2 = 1 \quad (39)$$

$$3G_0 + 15G_2 + 105G_4 = M_4/S^4 = M_4/M_2^2$$

This system is solved for  $G_0, G_2$  and  $G_4$ , the results are inserted into (26), and we can then calculate the approximate  $P^*$  by means of

$$P^* = \int_{-B_0}^{B_0} g(x_0) dx_0 = 2 \int_0^{B_0} g(x_0) dx_0$$

Using the general relation (38) again we find the following formula for  $P^*$

$$P^*(Z) = A(Z) - HZ(3 - Z^2)A'(Z) \quad (40)$$

where

$$H = \{3 - (M_4/M_2^2)\}/24 \quad (41)$$

$$Z = B_0/S \quad (42)$$

We shall call the estimate of  $P^*$  given by (40) the moment approximation.

## 6. NUMERICAL INTEGRATION OF THE CHARACTERISTIC FUNCTION

This procedure consists merely of carrying out the integration of (15), i.e., evaluating

$$P^* = \int_{-B_0}^{B_0} F^*(\chi_0) d\chi_0 = (2/\pi) \int_0^\infty \phi_0(t) t^{-1} \sin(B_0 t) dt \quad (43)$$

where  $\phi_0(t)$  is calculated from  $\phi_j(t)$  by (16). The  $\phi_j(t)$  are evaluated by use of (20) when  $\rho_j$  is of moderate size and (22) - (25) when  $\rho_j$  is large.

In general it is necessary to evaluate the integral by numerical means. To carry this out with sufficient accuracy is sometimes difficult because  $\phi_0(t)$  dies away in oscillatory fashion as  $t \rightarrow \infty$ . Usually, most of the contribution to the value of the integral comes from near  $t = 0$ , but significant contributions can also come from further out, where  $\phi_0(t)$  may oscillate rapidly. In cases where a significant contribution comes from the region of rapid oscillation, the integration must be carried out with great care.

The following procedure was adopted for carrying out these integrals. A fundamental range of  $t$  say  $T$ , was chosen, roughly small enough so that

$$\frac{d}{dt} [t^{-1} \phi_0(t) \sin(B_0 t)]$$

has no more than 5 zeroes in the range  $(n-1)T \leq t \leq nT$  for  $n = 1, 2, 3, \dots M$ .

M is taken so large that the ranges beyond  $n = M$  contribute negligibly to the integral. For each range of  $t$  the integral was evaluated by Gaussian Integration, and the total integral obtained by adding the results for all the ranges. Some experimentation was needed to find a suitable number of points to use in the Gaussian Integration and to determine how large  $M$  should be.

The computer program that carried out this evaluation of  $P^*$  was occasionally slow-running and was used primarily for spot-checking the results of the other methods.

## 7. SIMPLE APPROXIMATIONS AND LIMITING CASES

A number of obvious, simple approximations for  $P^*$  can be derived, and we describe three of them briefly here.

(i) If all the component tolerances are very large, i.e., if

$$D_j/\sigma_j = B_j/S_j \gg 1 \quad \text{for all } j$$

the distribution associated with each  $X_j$  is approximately normal with variance

$S_j^2$ . Then  $F^*(X_0)$  is approximately a normal distribution with variance  $S_I^2$ , where

$$S_I^2 = \sum_{j=1}^N S_j^2 \quad (44)$$

Hence we may write this estimate of  $P^*$  as a function of  $B_0$  in the form

$$P_I^*(B_0/S_I) = A(B_0/S_I) \quad (45)$$

(ii) A different approximation may be obtained if we assume that  $F^*(X_0)$  is a normal distribution with variance  $S_{II}^2$ , truncated at points  $|X_0| = \tau_0$  i.e.,

$$F^*(X_0) = 0 \quad |X_0| \geq \tau_0$$

In this case we find the following estimate

$$P_{II}^*(B_0/S_{II}) = A(B_0/S_{II})/A(\tau_0/S_{II}), \quad B_0 < \tau_0$$

$$= 1, \quad B_0 \geq \tau_0$$



The properties of the error function imply that

$$P_{II}^*(B_0/S_{II}) \geq P_I^*(B_0/S_I) \quad (46)$$

The accuracy of this approximation depends on how well  $F^*(X_0)$  is approximated by a truncated normal distribution and how precisely we can estimate  $\tau_0$ . There are two cases in which this approximation may be tolerably accurate. First, if one component, say the  $L$ -th, dominates the others, i.e.,  $S_L^2 \gg S_j^2$ ,  $j \neq L$  the output will be approximately that of the dominant component,  $X_L$ , which is a normal distribution truncated at  $\pm B_L$ . In this case we expect that  $P_{II}^*(B_0/S_{II})$  with  $\tau_0 = B_L$ , will be a fair approximation to  $P^*$ . Second, if all the  $C_j$  are roughly equal, and all the  $Z_j$  are roughly equal to  $Z_A$ , say, then we expect that  $F^*(X_0)$  will be approximately a normal distribution with variance  $S_I^2$  truncated at

$$\tau_0 = Z_A S_I$$

Hence in this case also  $P_{II}^*(B_0/S_{II})$  should be a decent approximation to  $P^*$ .

(iii) A third simple approximation is obtained by setting  $G_4 = 0$  in (26) and choosing  $G_0$  and  $G_2$  such that

$$\int_{-\infty}^{\infty} X_0^{2k} g(X_0) dX_0 = M_{2k} \quad \text{for } k=0, 1$$

Then we get as an approximation for  $P^*$  merely the first term of (42),

$$P_{III}^*(Z) = A(Z). \quad (47)$$

## 8. RESULTS

All the cases discussed here have four components, i.e.,  $N = 4$ , and all have  $\sigma_j = 1$ ,  $j = 1, 2, 3, 4$ . Eleven different cases were studied with various values for the  $D_j$  and  $C_j$  as shown in Table 1. For each case results were obtained in the form of graphs of  $P^*$  as a function of  $B_0/S_I$  and are displayed in Figures 1-11. Each graph shows the mean and standard deviation of the four repetitions of the Monte-Carlo Method as well as the moment approximation for that case. The values of  $S_I$ ,  $M_2 = S^2$  and  $M_4$  are listed in Table 2.

The method of integrating the Characteristic Function was used only at points where sizeable discrepancies were found between the Monte-Carlo results and the moment approximation. These points are shown on the appropriate graphs and compared with the other methods in Table 3.

Several additional graphs show comparisons among the moment method predictions for different cases. The comparison among Cases (I), (II) and (III) is shown in Figure 12, Cases (VI), (VII) and (VIII) in Figure 13 and Cases (IX), (X) and (XI) in Figure 14. Also, the simple approximation  $P_I^*$  is shown in Figure 12, and  $P_{II}^*$  in Figures 13 and 14 for relevant values of  $\tau_0$ .

## 9. DISCUSSION

We shall first compare and comment on the results obtained by the various methods, then suggest procedures for estimating  $P^*$  under various circumstances.

Figures 1 to 11 show that the agreement between the Monte-Carlo method and the moment-approximation is reasonably good in a general sense. We see from Table 3 that, when the results do not agree well, the integration of the characteristic function almost always agrees with the moment approximation.

Of the three methods one expects that integrating the characteristic function should be the most accurate. The Monte-Carlo method is usually thought to be somewhat inaccurate unless a very large number of samples is used, and the above results suggest that this is the case here. Procedures like the moment-approximation are fairly common in statistics and often give satisfactory accuracy. However, there are two theoretical defects of the moment-approximation here that are worth mentioning. First, the approximate density function  $g(X_0)$  is continuous (see Equation (26)) but the true density function,  $F^*(X_0)$ , is discontinuous. In fact

$$F^*(X_0) \equiv 0$$

when  $X_0 > \Delta$ . Second,  $g(X_0)$  is slightly negative for  $X_0$  sufficiently large in many cases. Neither of these defects seems to cause serious errors in the estimate of  $P^*$  for the cases studied here since the results agree well with the integration of the characteristic function. If serious errors are to arise, one would expect to find them when there is a single, dominant

component with a low tolerance on it, as in Case (VIII). Table 3 confirms this expectation, for we see that, when  $B_0/S_I = .945$ , the characteristic function and moment-approximation differ by .011, whereas the worst error observed in the other cases of Table 3 is only .001. However, even in this most unfavorable case the error in the moment-approximation is small enough to be unimportant.

Figure 12 shows how curves for  $P^*$  change as the common tolerance value for the four components increases from  $B_j/S_j = 1$  through 1.5 to 2. As we expect, the curves become lower and tend toward the normal curve, given by  $P_I^*$ , with increasing component tolerances.

The effect on  $P^*$  of an increasingly dominant component is displayed in Figures 13 and 14. Figure 13 shows the case where the increasingly dominant component has a smaller tolerance than the other components. As  $C_1$  increases from 1 through 2 to 5, the curve of  $P^*$  is raised toward the curve for  $P_{II}^*$  truncated at  $B_0/S_I = 1$ , to which it must ultimately tend. In contrast Figure 14 shows what happens when the increasingly dominant component has a higher tolerance than the others. As  $C_1$  increases from 1 through 2 to 5, the curve for  $P^*$  is lowered toward the curve for  $P_{II}^*$  truncated at  $B_0/S_I = 2$ , to which it ultimately tends.

When all the  $C_j$  are roughly the same, we may also inquire about the effect of changing the component tolerances but keeping the average component tolerance constant. Comparing Cases (II), IV) and (V) we see that this has

scarcely any effect on  $P^*$ . In other words, when the  $C_j$  are roughly equal, the mean component tolerance has a considerable influence on  $P^*$  (see Figure 12) but the variance in the component tolerances has negligible effect.

A reasonably extensive comparison of the simple approximations, (47), (48) and (49) with the more accurate calculations suggests the following as a rule:

(i) Use  $P_{II}^*$  if one component dominates greatly.

(ii) Use  $P_I^*$  if  $B_j/S_j \geq 2$  for all  $j$ .

(iii) Use  $P_{III}^*$  if no one component dominates.

Use of this rule will give fair results, perhaps suitable for an initial estimate. The simplest accurate procedure is the moment-approximation, given by (42) - (44).

#### 10. WORKED EXAMPLE

A certain piece of mechanical equipment is supposed to operate at a load of 900 lbs. Three components, standard items, are assembled to form this mechanism. Components 1 and 3 are springs and component 2 is an electrical switch. Their component errors,  $Y_1$ ,  $Y_2$  and  $Y_3$ , are related to the error in the output load by (1) where

$$C_1 = 240 \text{ lb/inch}$$

$$C_2 = 15 \text{ lbs/volt}$$

$$C_3 = 210 \text{ lb/inch}$$

From information about the manufacture of the components we know that the distribution of their errors is roughly normal with zero mean and

#1: 80% are acceptable at a tolerance of .1-inch

#2: 70% are acceptable at a tolerance of 1 volt

#3: 92% are acceptable at a tolerance of .1-inch

From Tables of the normal distribution we find the standard deviation  $\sigma_j$  as follows:

#1: .8 acceptable corresponds to .1 = 1.28  $\sigma_1$

$$\sigma_1 = .0781\text{-inches}$$

#2: .7 acceptable corresponds to 1 = 1.04  $\sigma_2$

$$\sigma_2 = .962 \text{ volts}$$



#3: .92 acceptable corresponds to  $.1 = 1.75 \sigma_3$

$$\sigma_3 = .0571 \text{ inches}$$

Then

$$S_1 = 240 \times .0781 = 18.7 \text{ lbs.}$$

$$S_2 = 15 \times .962 = 14.4 \text{ lbs.}$$

$$S_3 = 210 \times .0571 = 12.0 \text{ lbs.}$$

The tolerances established on the components are

$$D_1 = .15 \text{ inches, } D_2 = 2 \text{ volts, } D_3 = .07\text{-inches}$$

and therefore from (4) and (12)

$$B_1 = 36 \text{ lbs, } B_2 = 30 \text{ lbs, } B_3 = 14.7 \text{ lbs.}$$

$$\Delta = 80.7 \text{ lbs.}$$

The tolerance on the load needed to activate the mechanism is

$$B_0 = 30 \text{ lbs.}$$

Since  $B_0 < \Delta$  we know that  $0 < P^* < 1$ , i.e., we know it is possible that each component will satisfy its tolerance but the tolerance on the load will be violated.

First we find a quick, rough estimate of  $P^*$ . In order to determine which one of the simple estimates, (45) - (47), is best, we find from (37)

$$Z_1 = B_1/S_1 = 1.93, \quad Z_2 = 2.08, \quad Z_3 = 1.22$$

These are not all  $\geq 2$ . Also none of the  $C_j$  dominates all the others. The rule stated at the end of section 9 suggests, therefore, that we use  $P_{III}$ . To find  $P_{III}$  we need to find  $M_2$  by means of (32) and (35). From the tables of the normal function we find

$$\begin{aligned} A(Z_1) &= .946, & A(Z_2) &= .962, & A(Z_3) &= .778 \\ A'(Z_1) &= .120, & A'(Z_2) &= .092, & A'(Z_3) &= .379 \end{aligned}$$

Putting these values into (35), then combining the results with (32) we get

$$M_2 = 480, \quad S = 21.9$$

From (42)  $Z = 30/21.9 = 1.37$ . Using the estimate (47), the normal function table gives

$$P_{III}^* = A(1.37) = .829$$

The more accurate approximation (40) involves finding  $M_4$  in addition to  $M_2$ . The calculation of  $M_4$  by (34) and (36) leads to  $M_4 = 625,000$ . Putting this into (41) and combining with (40) we find

$$P^*(Z) = .829 - .006 = .823$$

Thus, if all components satisfy their tolerances, the probability that the mechanism will operate at a load between 870 and 930 lbs. is about .82. On the average, therefore, if we need 1,000 workable mechanisms, we should

expect to assemble about

$$1000/.823 = 1215$$

out of satisfactory components.

#### REFERENCES

1. Bowker, A. H., and Lieberman, G. J., "Engineering Statistics", Prentice Hall, Englewood Cliffs, N. J., 1959, Chapter 3.
2. Abramowitz, M., and Stegun, I. A., "Handbook of Mathematical Functions", National Bureau of Standards, Applied Math Series No. 55, Sixth Printing, November 1967.

# APPENDIX A - FORMULAS RELATING TO THE ERROR FUNCTION

The following are fundamental formulas relating to the Error Function and are taken from Reference <sup>2</sup>.

$$\operatorname{erf}(Z) = 2\pi^{-1/2} \int_0^Z e^{-u^2} du \quad (\text{A.1})$$

In (A.1) the integration may be carried out along any path in the complex U-plane connecting U=0 and U=Z. An alternative definition is

$$A(Z) = \operatorname{erf}(Z) = (2\pi)^{-1/2} \int_{\lambda=-Z}^Z e^{-\lambda^2/2} d\lambda \quad (\text{A.2})$$

We have also

$$\operatorname{erf}(-Z) = -\operatorname{erf}(Z) \quad (\text{A.3})$$

$$\operatorname{erf}(\bar{Z}) = \overline{\operatorname{erf}(Z)} \quad (\text{A.4})$$

where the bar denotes the complex conjugate.

Various derivatives of these quantities can be found from

$$\frac{d^{k+1}}{dZ^{k+1}} (\operatorname{erf} Z) = (-1)^k 2\pi^{-1/2} e^{-Z^2} H_k(Z), \quad (\text{A.5})$$

$$K \geq 0$$

$$A'(Z) \equiv dA/dZ = (2/\pi)^{1/2} e^{-Z^2/2} \quad (\text{A.6})$$

where  $H_k(Z)$  is the Hermite Polynomial of k-th degree. These satisfy the relations

$$H_0(z) = 1, \quad H_1(z) = 2z \quad (A.7)$$

$$H_{k+1}(z) = 2z H_k(z) - 2k H_{k-1}(z). \quad (A.8)$$

Also we list several formulas involving the complex function  $W$ .

$$W(z) \equiv W_r(z) + iW_i(z) = e^{-z^2} [1 - \operatorname{erf}(-iz)] \quad (A.9)$$

or

$$\operatorname{erf}(z) = 1 - W(iz)e^{-z^2} \quad (A.10)$$

and

$$W(\bar{z}) = \overline{W(-z)} \quad (A.11)$$

$W(z)$  may be found from the rational approximation

$$W(z) = iz \sum_{k=1}^3 r_k / (z^2 - \eta_k) + \epsilon(z) \quad (A.12)$$

provided  $W_r > 3.9$  or  $W_i > 3$ . The error  $\epsilon(z)$  satisfied the inequality  $|\epsilon(z)| \leq 2 \times 10^{-6}$ . The constants  $r_k$  and  $\eta_k$  have the following values

$$r_1 = .4613135, \quad r_2 = .09999216, \quad r_3 = .002883894$$

$$\eta_1 = .1901635, \quad \eta_2 = 1.7844927, \quad \eta_3 = 5.5253437.$$

TABLE 1

Case	$C_1$	$C_2$	$C_3$	$C_4$	$D_1$	$D_2$	$D_3$	$D_4$
I	1	1	1	1	1	1	1	1
II	1	1	1	1	1.5	1.5	1.5	1.5
III	1	1	1	1	2	2	2	2
IV	1	1	1	1	1	1.5	1.5	2
V	1	1	1	1	1	1	2	2
VI	1	1	1	1	1	1.5	1.5	1.5
VII	2	1	1	1	1	1.5	1.5	1.5
VIII	5	1	1	1	1	1.5	1.5	1.5
IX	1	1	1	1	2	1	1	1
X	2	1	1	1	2	1	1	1
XI	5	1	1	1	2	1	1	1

Component Error Coefficients ( $C_j$ ) and Tolerances ( $D_j$ ) in the Eleven Cases Studied

TABLE 2

Case	$S_I$	$M_2$	$M_4$
I	2	1.165	3.709
II	2	2.206	13.53
III	2	3.095	27.22
IV	2	2.168	13.10
V	2	2.130	12.67
VI	2	1.946	10.47
VII	2.646	2.819	21.60
VIII	5.292	8.933	182.5
IX	2	1.647	7.490
X	2.646	3.968	40.90
XI	5.292	20.22	988.5

The Standard Deviation,  $S_I$ , of the Output Distribution Calculated by the Normal Relation, and the Output Moments  $M_2$ ,  $M_4$ , in the Eleven Cases Studied



TABLE 3

Case	$B_0$	$B_0/S_I$	P* Monte Carlo	P* Char. Function	P* Mom. App.
(I)	1.6	.8	.822	.858	.858
(III)	2.8	1.4	.860	.887	.837
(IV)	2.4	1.2	.870	.896	.896
(V)	1.2	.6	.603	.579	.580
(V)	2	1.0	.849	.825	.825
(VI)	1.2	.6	.612	.599	.600
(VIII)	5	.945	.928	.915	.904
(IX)	1.2	.6	.677	.639	.640
(IX)	2.4	1.2	.912	.940	.940
(X)	3.175	1.2	.907	.886	.886

Comparison among the Values of P\* given  
by the Three Methods of Calculation for  
Various Cases and Tolerances

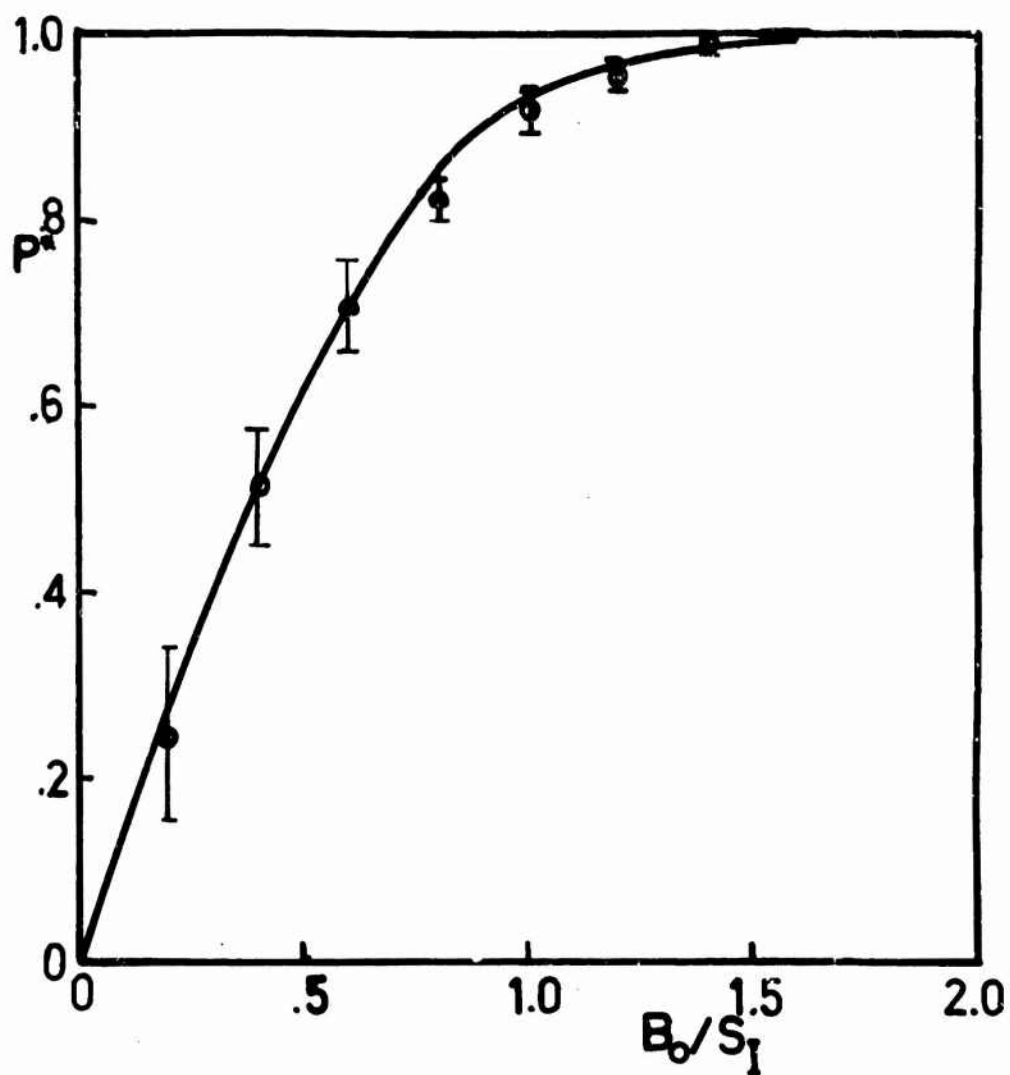


Figure 1:  $P^*(B_0/S_I)$  in Case I. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

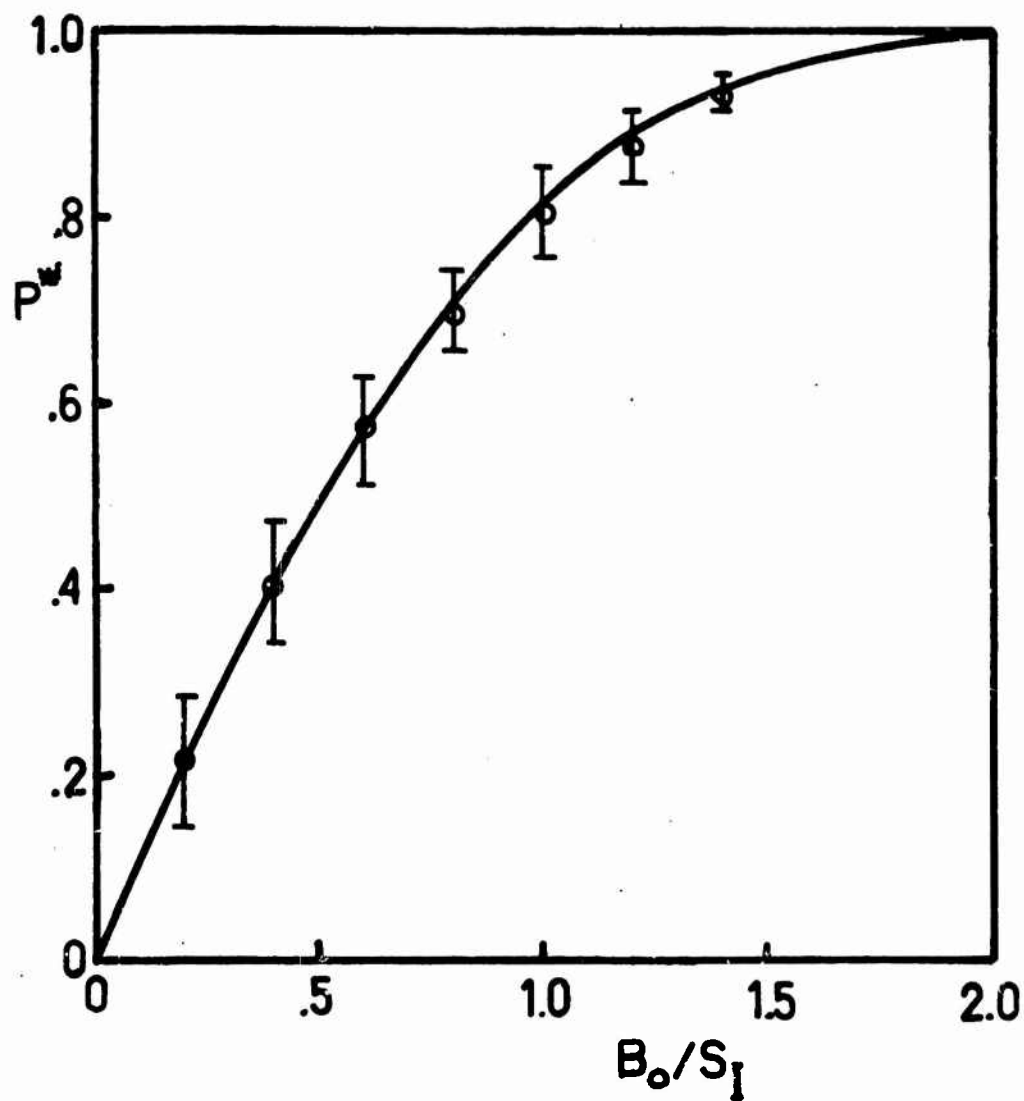


Figure 2:  $P^*(B_0/S_I)$  in Case II. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

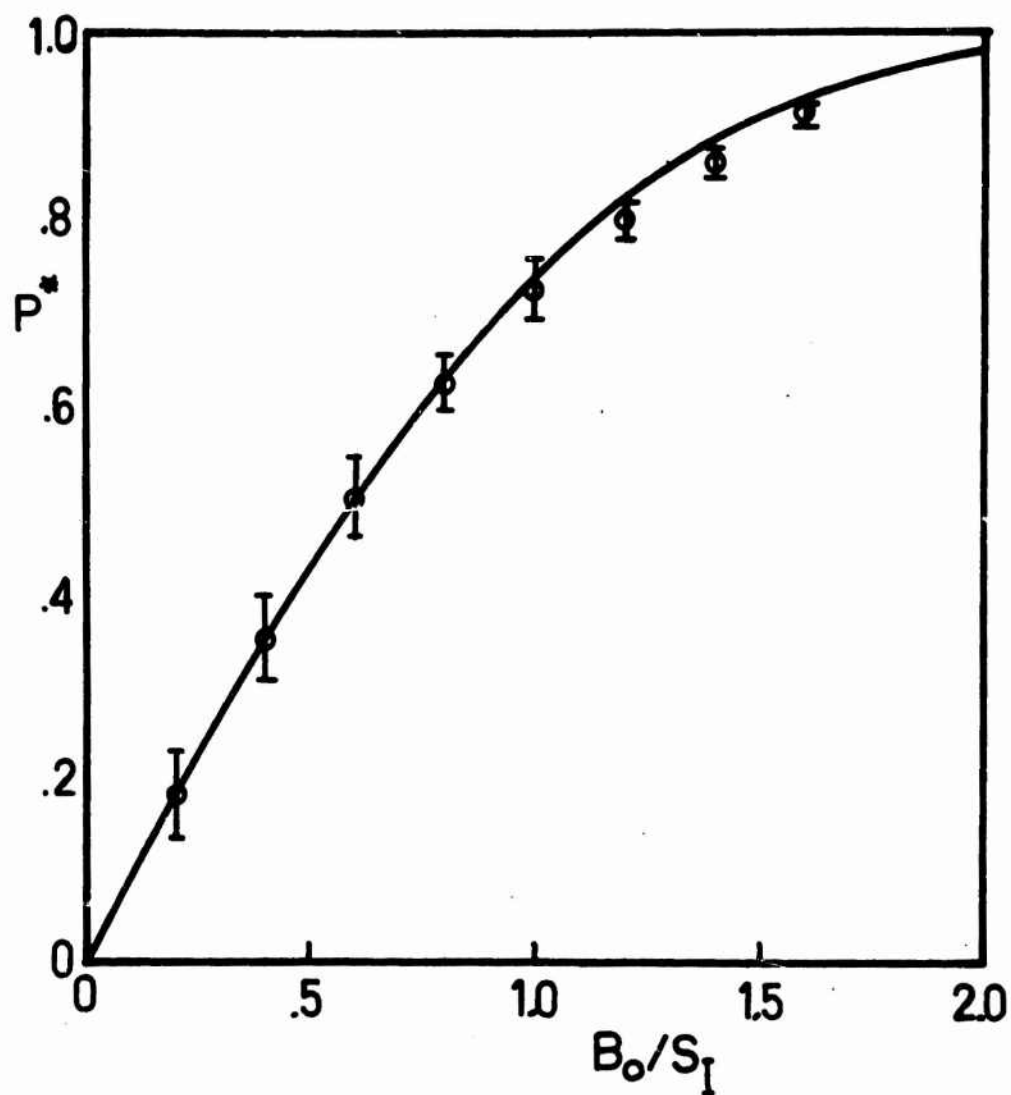


Figure 3:  $P^*(B_0/S_1)$  in Case III. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

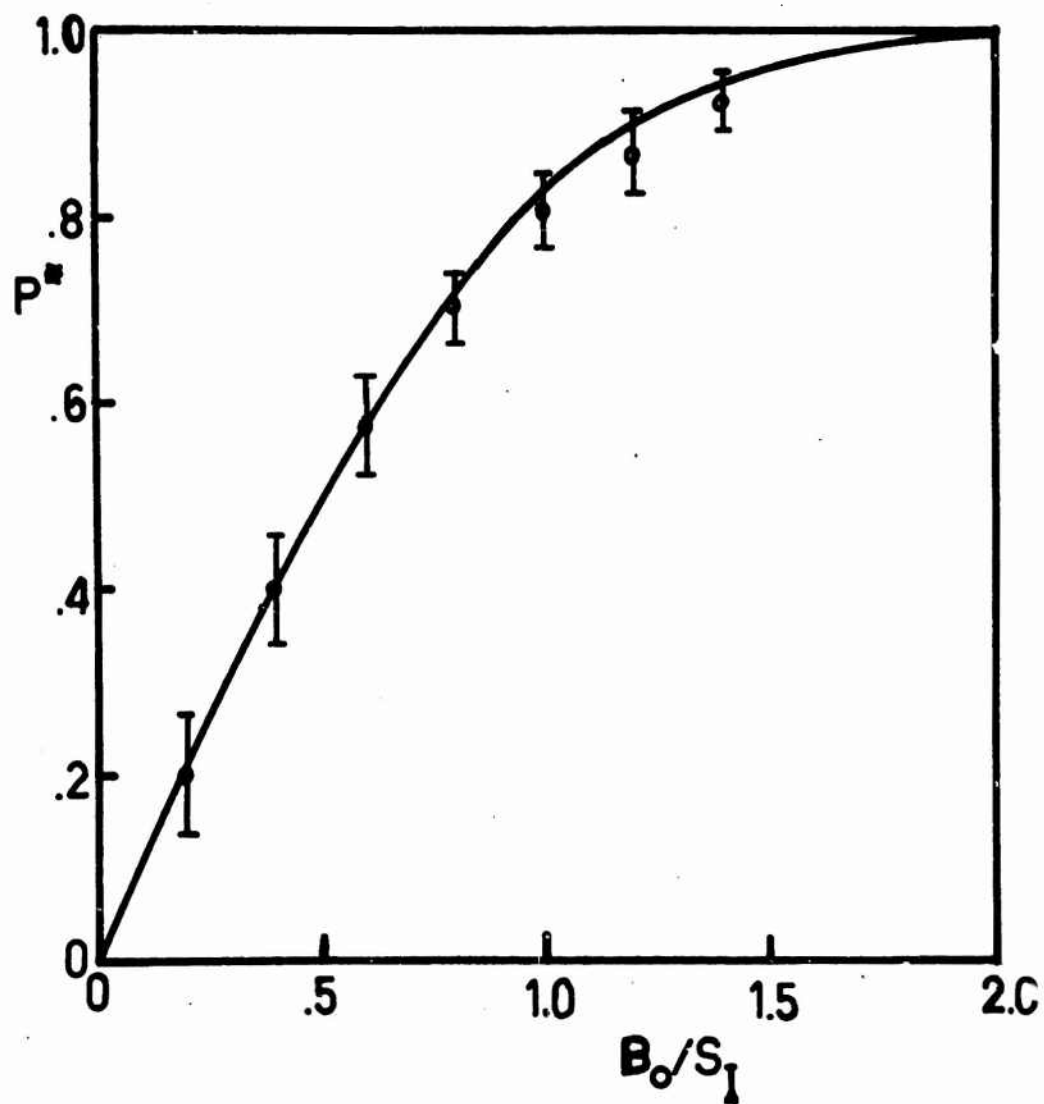


Figure 4:  $P^*(B_0/S_1)$  in Case IV. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

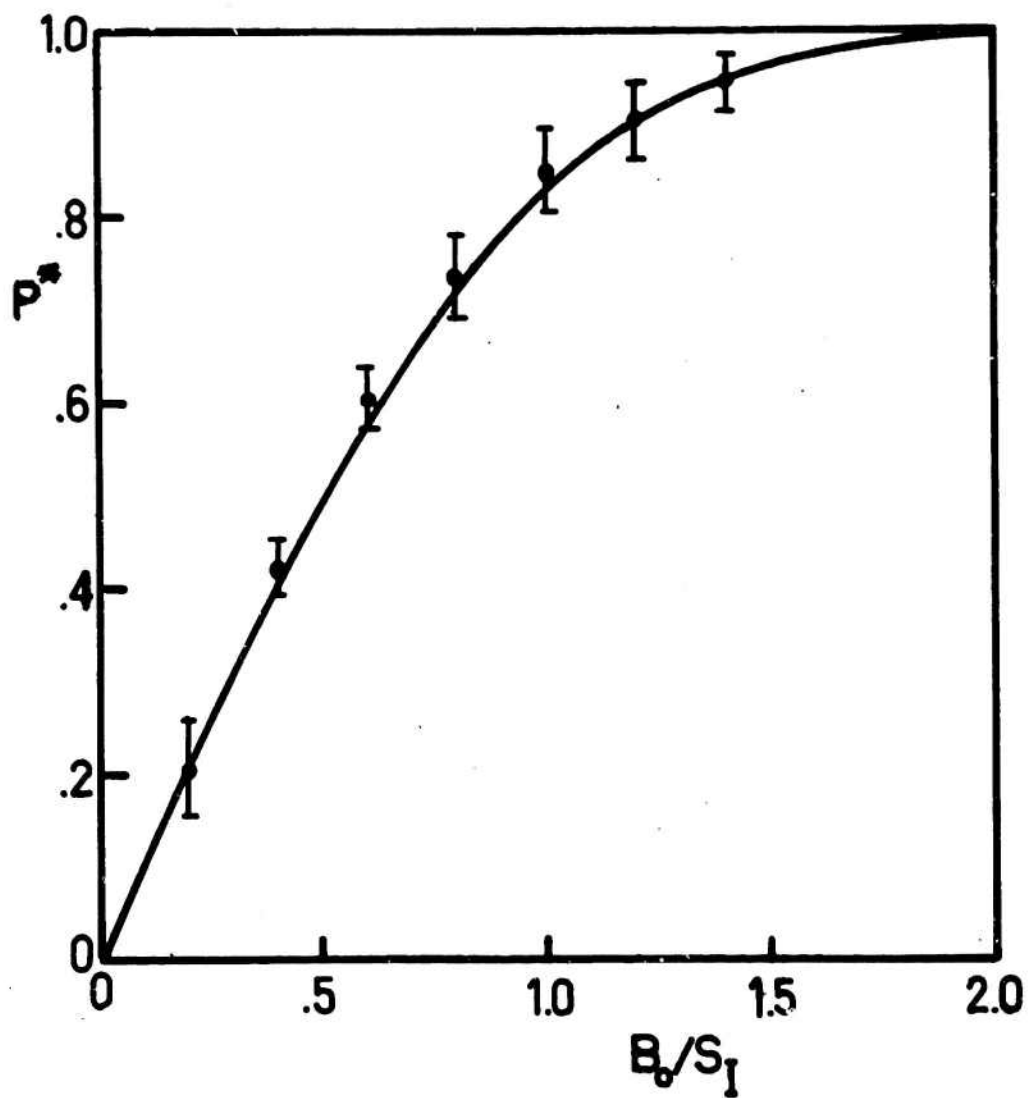


Figure 5:  $P^*(B_0/S_I)$  in Case V. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

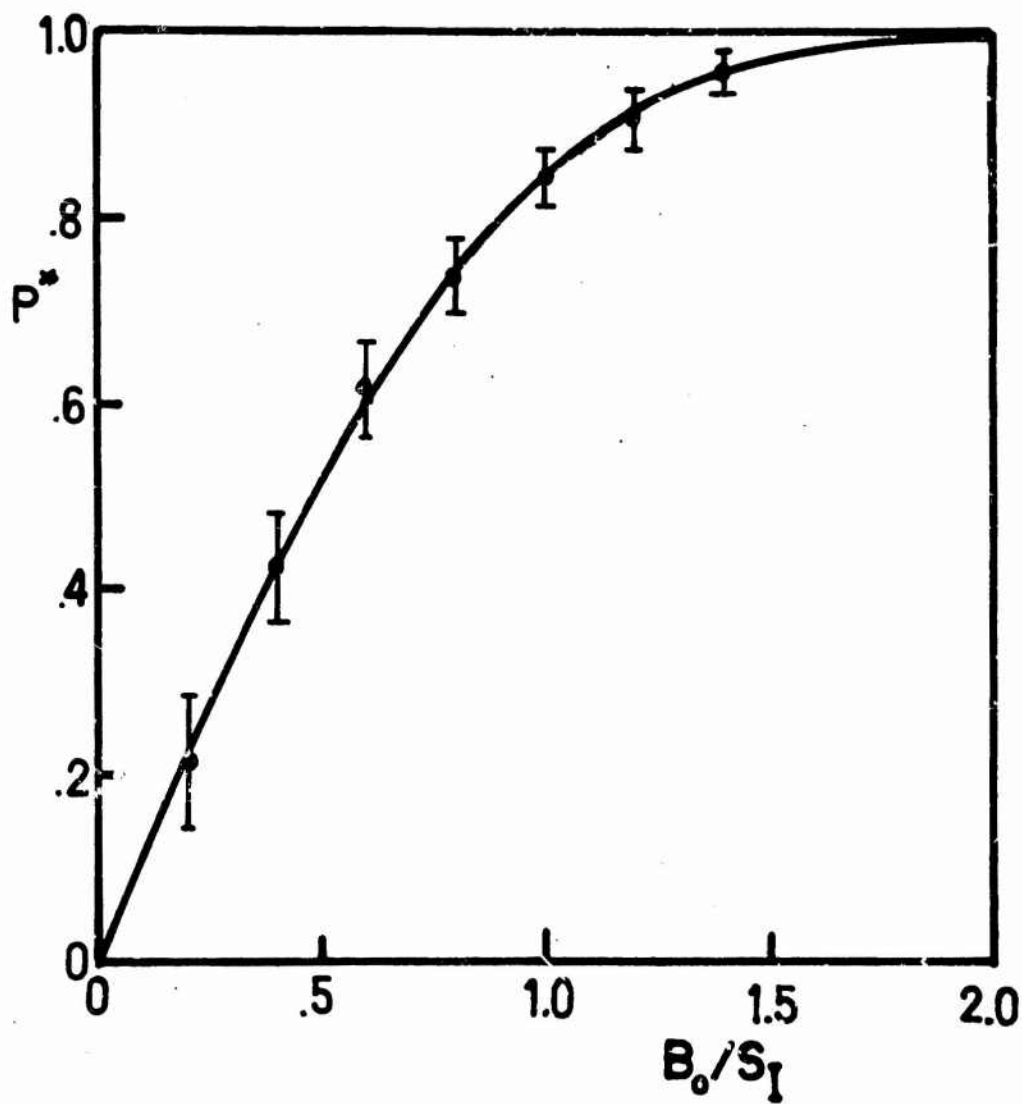


Figure 6:  $P^*(B_0/S_1)$  in Case VI. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

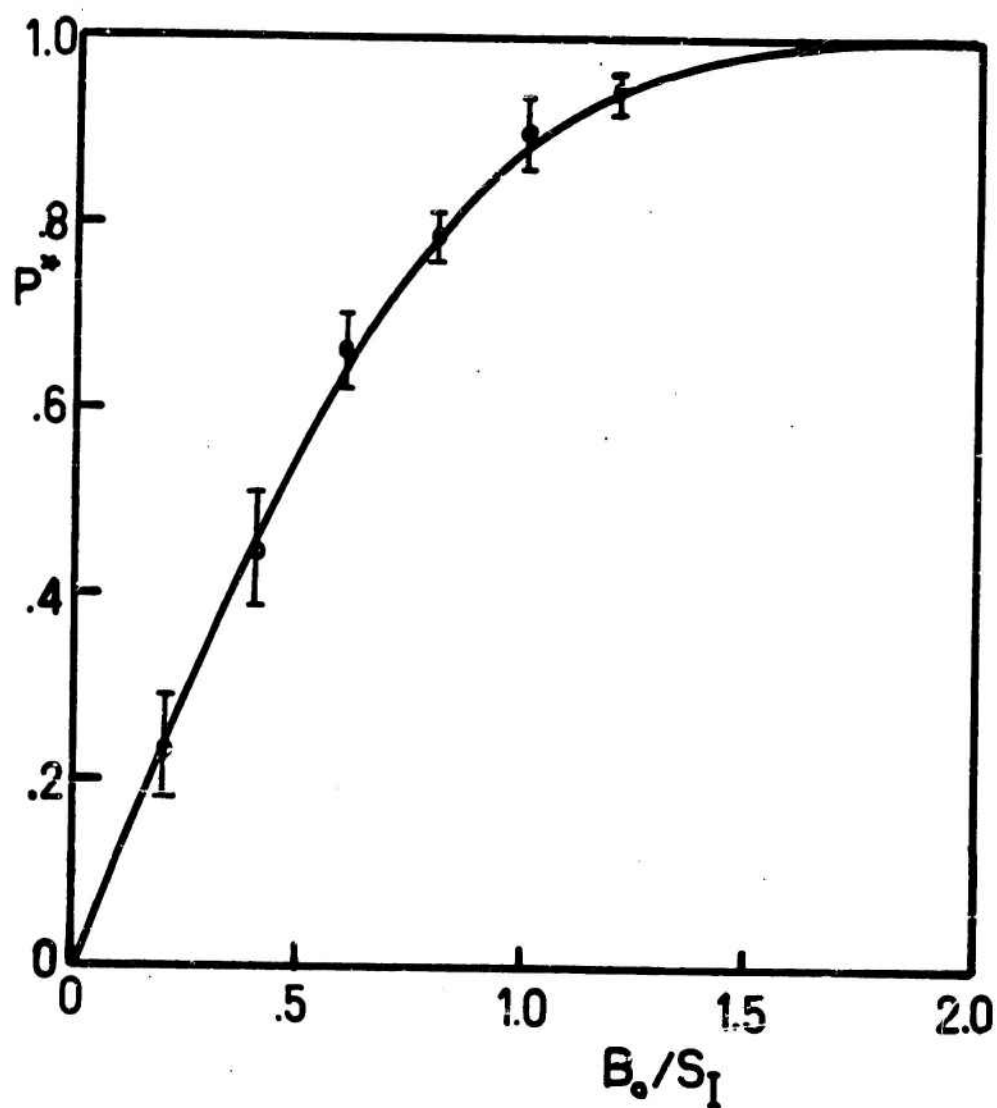


Figure 7:  $P^* (B_0/S_I)$  in Case VII. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates



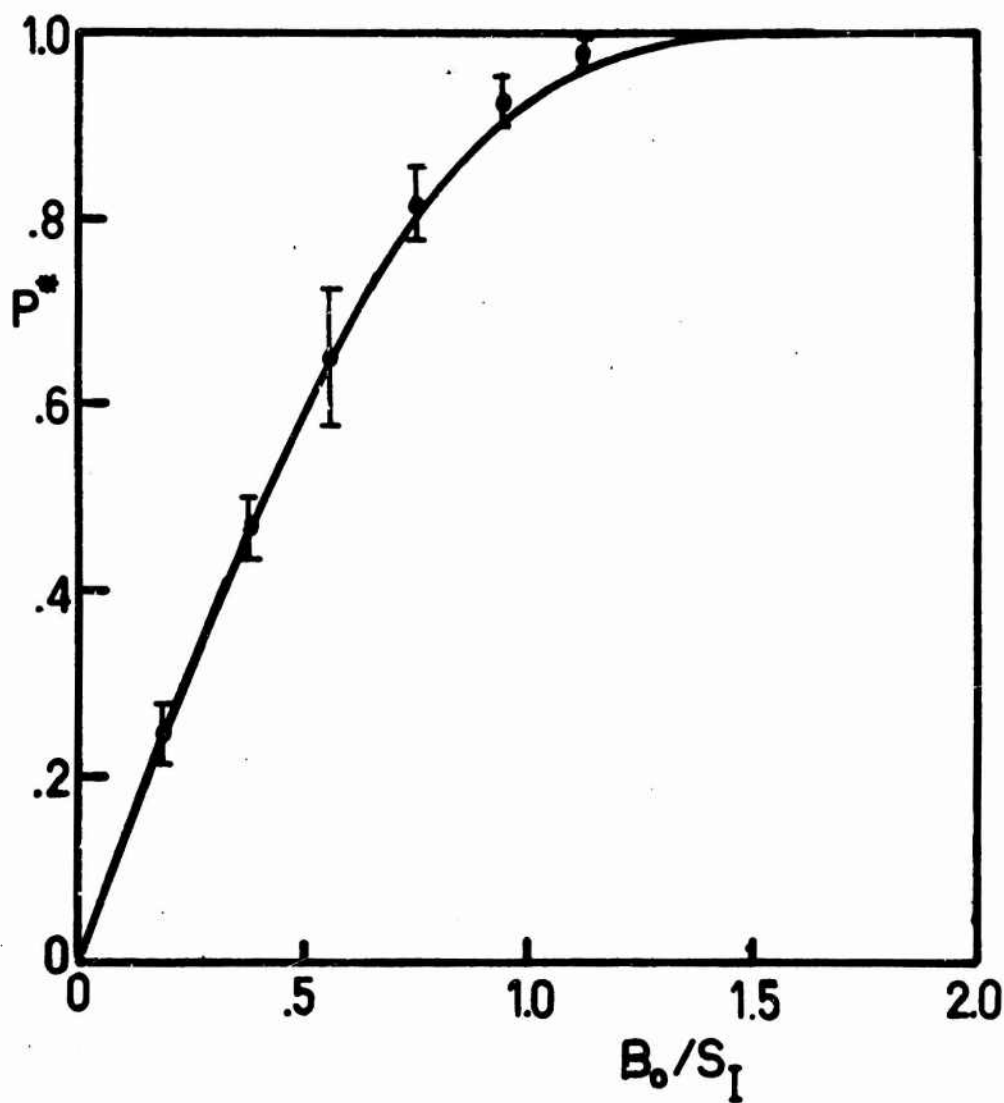


Figure 8:  $P^* (B_0/S_I)$  in Case VIII. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

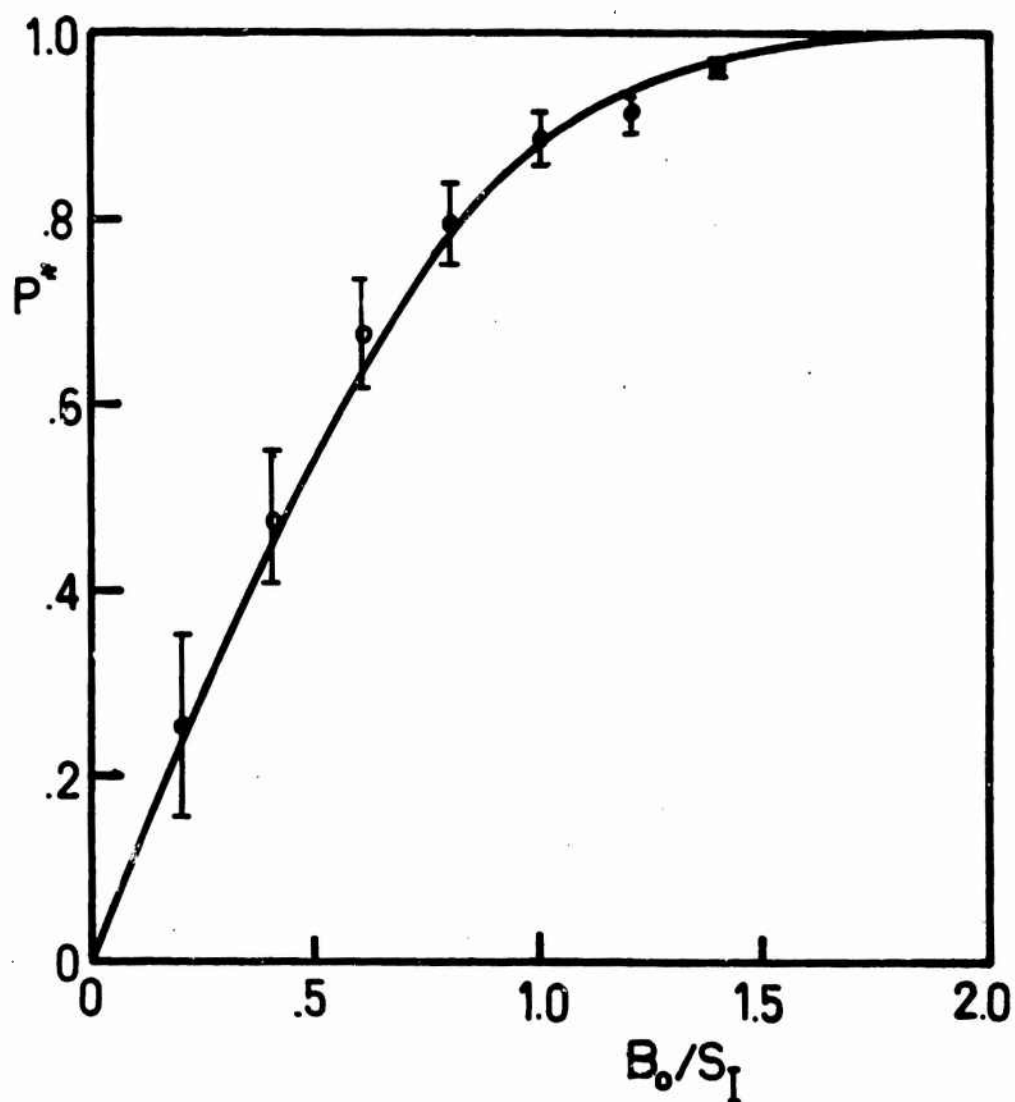


Figure 9:  $P^*(B_0/S_I)$  in Case IX. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

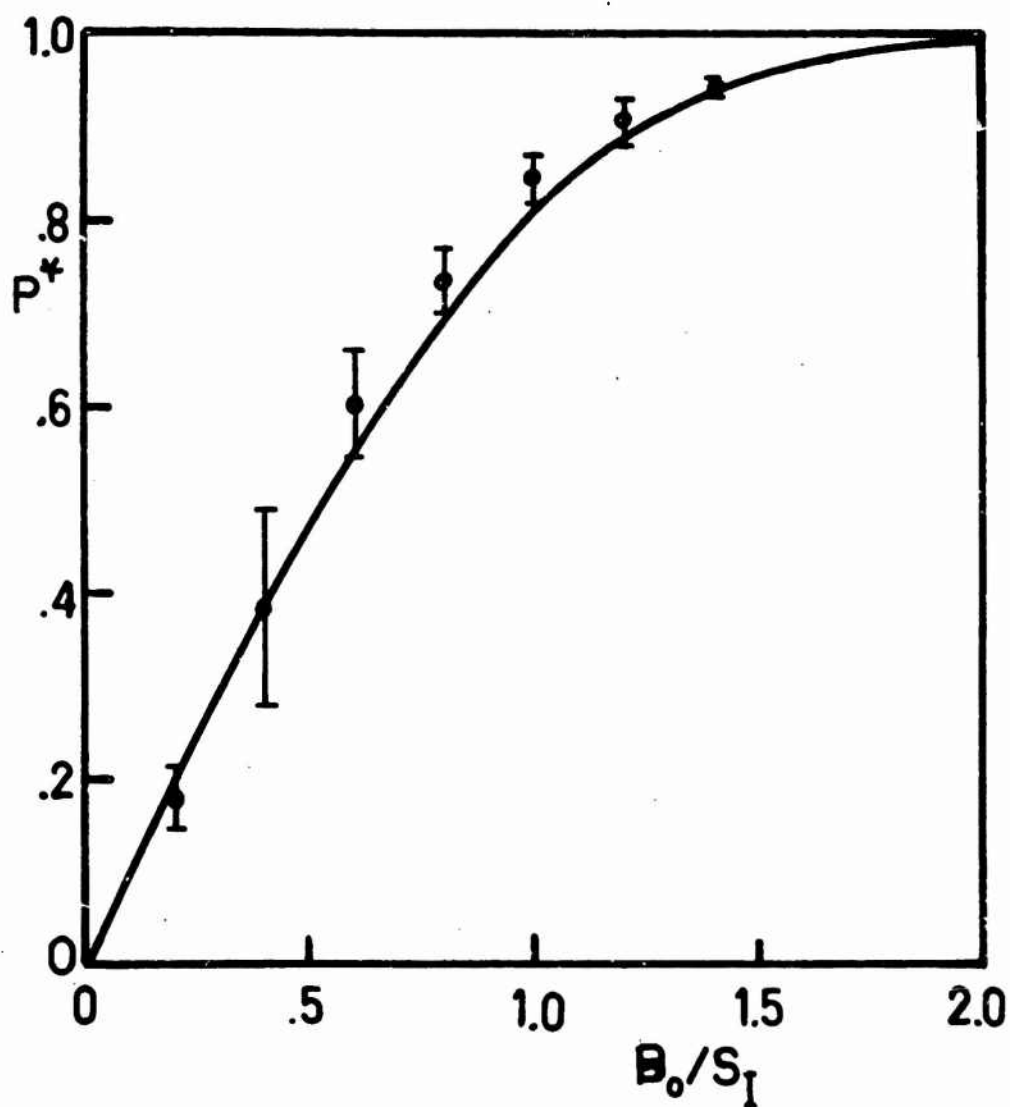


Figure 10:  $P^*(B_0/S_I)$  in Case X. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

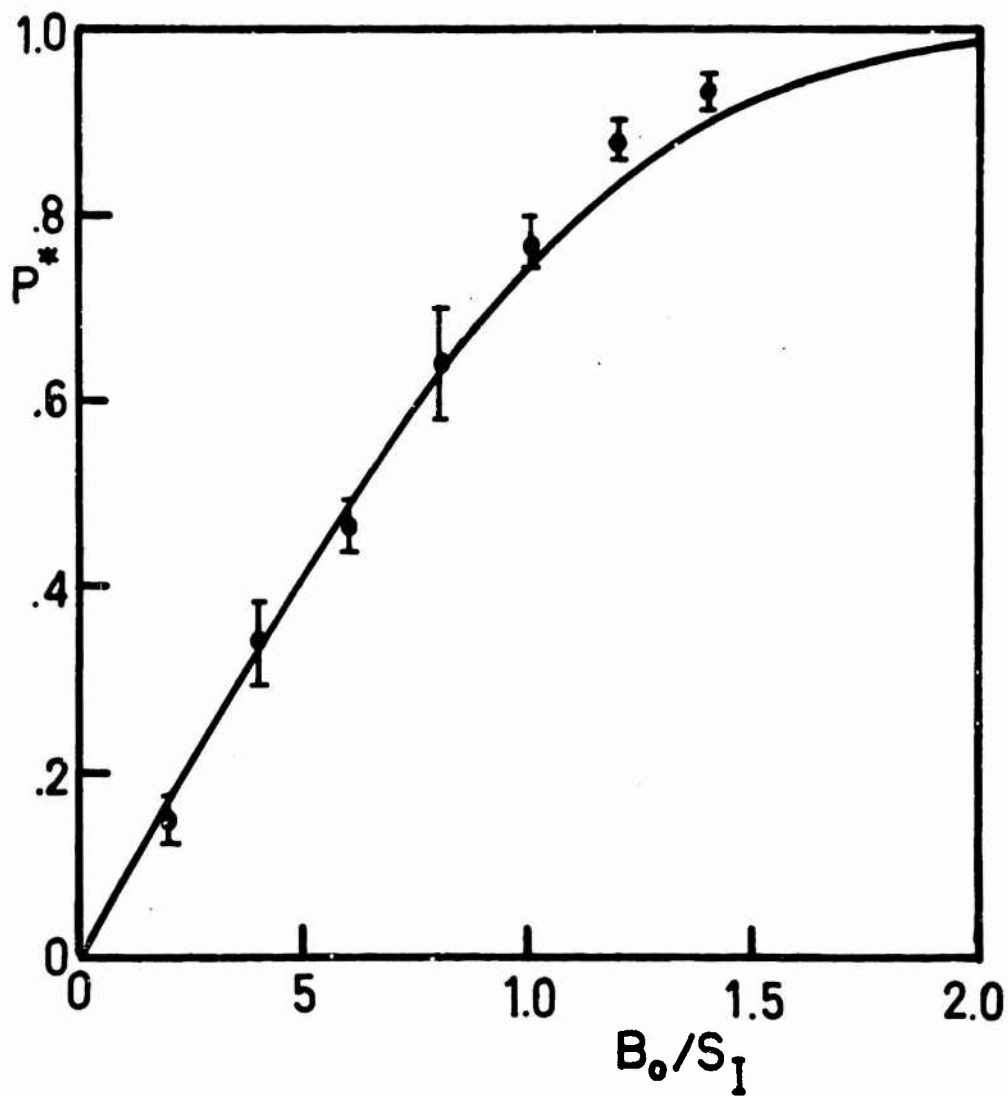


Figure 11:  $P^*(B_0/S_1)$  in Case XI. The curve is the moment approximation, (40), and the circles are Monte-Carlo estimates

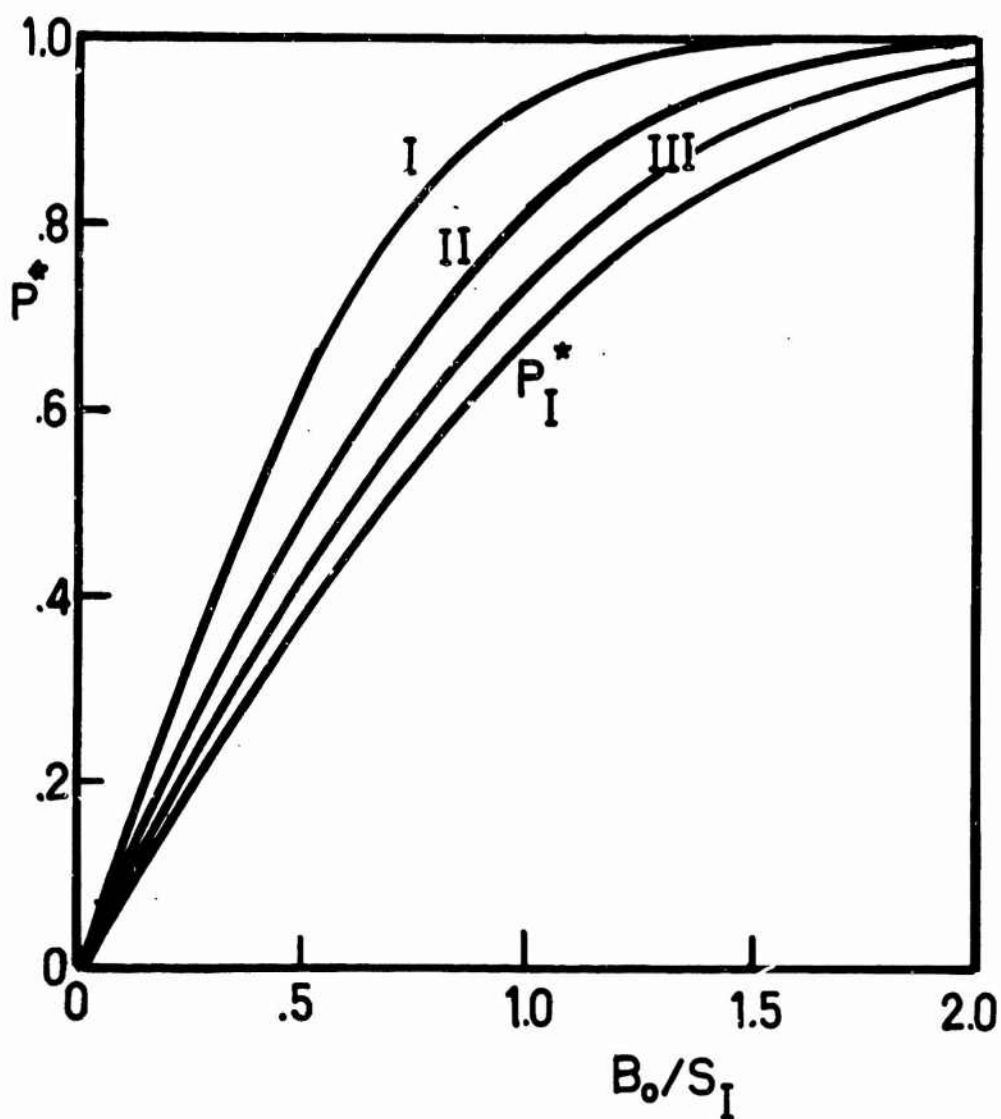


Figure 12: Comparison of  $P^*$  ( $B_0/S_I$ ) for Cases (I), (II), (III), using moment prediction, and  $P_I^*$ .

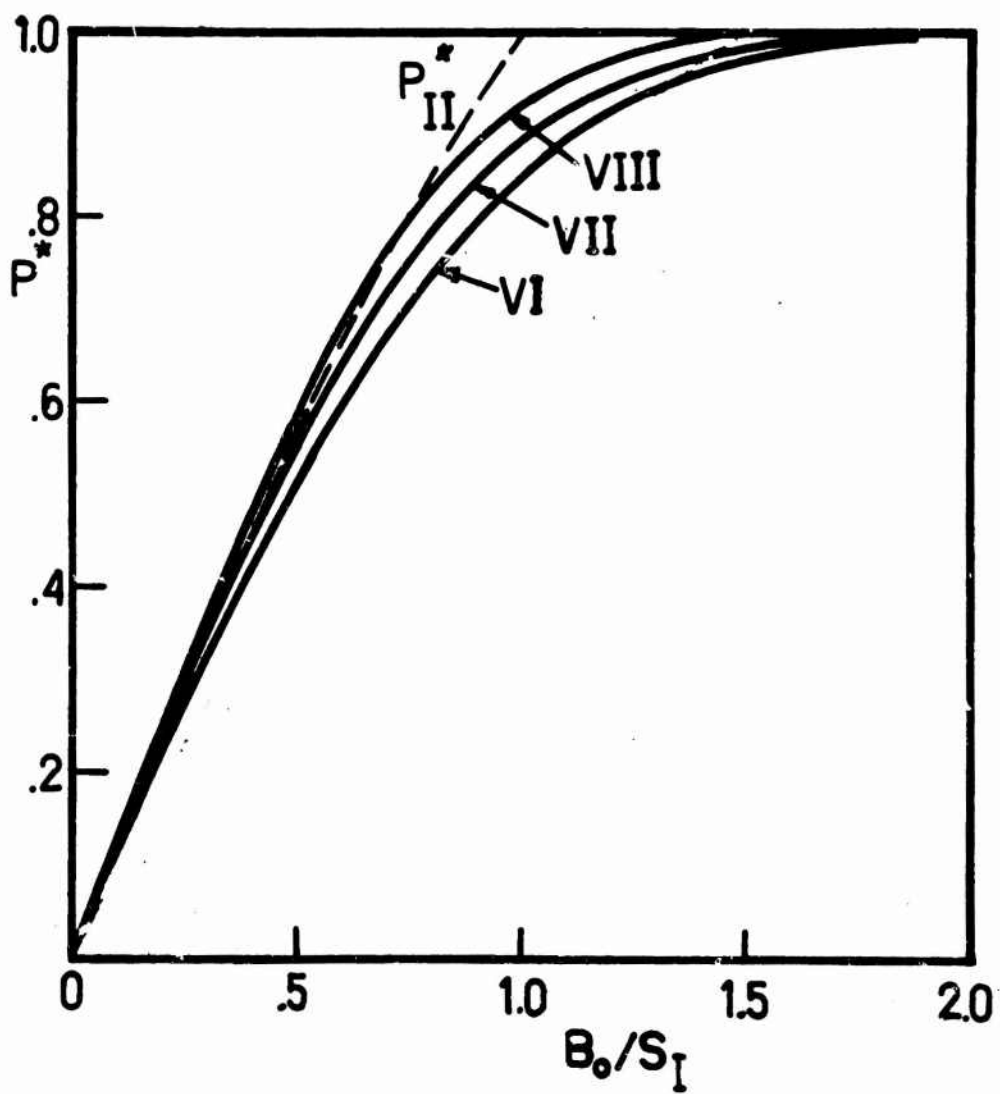


Figure 13: Comparison of  $P^*$  ( $B_0/S_I$ ) for Cases (VI), (VII), (VIII), using moment prediction, and  $P_{II}^*$ .

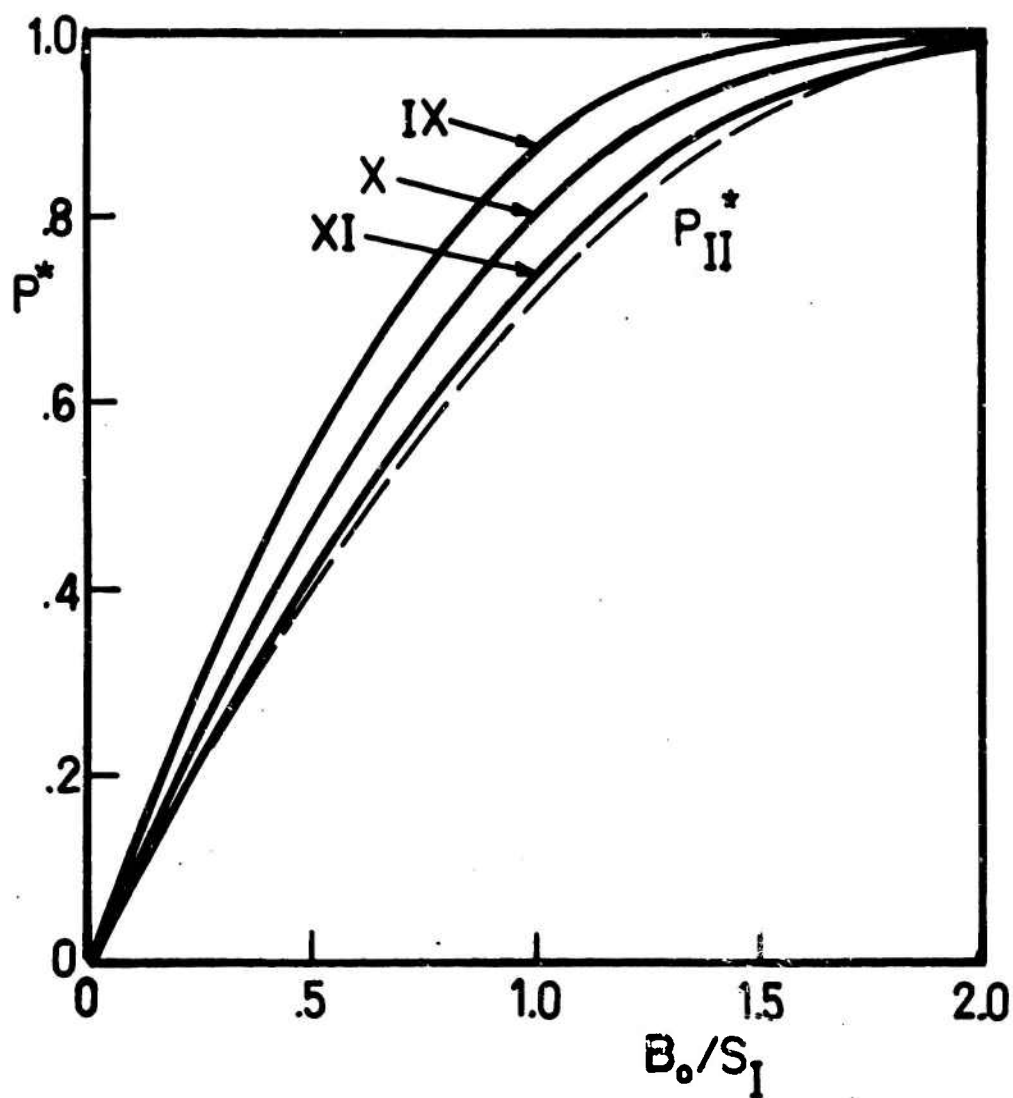


Figure 14: Comparison of  $P^*$  ( $B_0/S_1$ ) for Cases (IX), (X), (XI), using moment prediction, and  $P_{II}^*$ .

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<p>This report describes an investigation of how errors in components of an assembly can affect its performance. In particular the report deals with the situation, uncommon in engineering practice, where the output tolerance of the assembly may be violated even though the tolerances on the components are all met. This situation is analyzed to estimate the probability that the output tolerance will be satisfied given that the component tolerances are met. Three methods are described for estimating this probability, their results are compared in a number of cases, and a best method is chosen. Several simple rules, suitable for preliminary estimates, are also given. An example is worked out showing a simple application of the method.</p>		

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Tests	9					
Equipment	9					
Components			6			
Output			7			
Performance			7			
Probability			8,9			
Estimating			8			
Methods			8			

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